Neoclassical Growth Model

$Y = F[K, L, A]$

Assumption
① $F[\lambda K, \lambda L, A] = \lambda F[K, L, A]$
② $\frac{\partial F}{\partial K} > 0 \quad \frac{\partial^2 F}{\partial K^2} < 0$
③ Inada Condition 稲田条件
$\lim_{K \to 0} \frac{\partial F}{\partial K} = \infty \quad \lim_{K \to \infty} \frac{\partial F}{\partial K} = 0$

Property
① $MPK[\lambda K, \lambda L, A] = MPK[K, L, A]$
② $F[K, L, A] = K \cdot MPK + L \cdot MPL$
③ $F[K, 0, A] = 0 \quad F[0, L, A] = 0$
④ Unboundedness $\lim_{K \to \infty} F[K, L, A] = \infty$

Harrod-Neutral $\to Y = F[K, AL]$
$+$ NA.
$\Rightarrow$ Solow Model

$\boxed{\text{Assumption}}$ → Behavior → Equilibrium Path & S.S. → Shock.

A on Production

$$\begin{cases} Y = F[K, AL] \\ F[\frac{K}{AL}, 1] = \frac{1}{AL} F[K, AL] = Y/AL \\ \text{令 } y = f(k) \quad k = \frac{K}{AL} \quad f(k) = F(\frac{K}{AL}, 1) = F[k, 1] \quad y = \frac{Y}{AL} \end{cases}$$

Neoclassical · A ·

① $\underline{f'(k) > 0}$ $\quad MPK[K, AL] \stackrel{def}{=} \frac{\partial F}{\partial K} = \frac{\partial [AL \cdot f(k)]}{\partial k \cdot AL}$

$$= AL \cdot f'(k) \frac{1}{AL} > 0$$

② $f''(k) < 0$

③ $MPL(K, AL) = \frac{\partial F(K, AL)}{\partial L} = \frac{\partial [AL \cdot f(k)]}{\partial L} = \frac{\partial [AL \cdot f(\frac{K}{A} \cdot \frac{1}{L})]}{\partial L}$

$$= Af(k) + ALf'(k)(\frac{K}{A})(-\frac{1}{L^2})$$

$$= A[f(k) - k f'(k)] > 0$$

$$\frac{\partial F[K, AL]}{\partial AL} = f(k) - k f'(k) > 0$$ marginal product of effective labor

④ $\lim_{k \to 0} f'(k) = \infty \quad \lim_{k \to \infty} f'(k) = 0$

⑤ $f(0) = 0$

"变量的增长率" $\quad \dot{Y} \stackrel{def}{=} \frac{dY}{dt}$

$$\dot{A}(t) = gA(t)$$

$$\frac{\dot{A}(t)}{A(t)} = \frac{dA/dt}{A(t)} = \frac{dA(t)/A(t)}{dt} = \frac{d \ln A(t)}{dt} = g$$

$$\ln A(t) = gt + C \quad A(t) = e^{gt} \cdot e^{C} \quad A(0) = e^{C}$$

$$\Rightarrow A(t) = A(0) e^{gt} \qquad L(t) = L(0) e^{nt}$$

A on Endowment $L(0)$ $K(0)$ $A(0)$ are given

$\dot{L}(t) = nL(t)$ $L(t) = L(0)e^{nt}$

A on Preference No

A on Market Structure 完全竞争 $\begin{cases} \text{Final good} \\ \text{Labor} \\ \text{Capital} \end{cases}$

Behavior

Household 外生

all 劳动力

储蓄 → 资本形成

消费

Firm

maxmize profits

$$\max_{K,L} F(K,AL) - RK - wL$$

rent rate

real wage

$I = S \cdot \text{Total income}$

$\dot{K} = I - \delta K$

折旧率

$K_{t+1} - K_t = I - \delta K_t$

F.O.C $\begin{cases} F_K(K,AL) = R \\ F_L(K,AL) = w \end{cases}$

Equilibrium Path

$I = S \cdot \text{Total income}$

$\dot{K} = I - \delta K$ Y

$\Rightarrow \dot{K} = I - \delta K = sF(K,AL) - \delta K$

$$\begin{cases} Y(t) = F[K(t), A(t)L(t)] \\ C(t) = (1-s)Y(t) \\ R(t) = F_K[\ ] \\ w(t) = F_L[\ ]. \end{cases}$$

Steady State:

$$\frac{\dot K}{K} = \frac{sF(K, AL) - \delta}{K} \qquad \underline{K = AL\cdot k} \qquad \frac{\dot k}{k} = \frac{\dot K}{K} - \frac{\dot A}{A} - \frac{\dot L}{L} = \frac{\dot K}{K} - g - n$$

$$\frac{\dot K}{K} = \underbrace{\frac{\dot A}{A}}_{g} + \underbrace{\frac{\dot L}{L}}_{n} + \frac{\dot k}{k}$$

$$\Downarrow$$

$$\frac{s\,ALf(k)}{K} - \delta = s\frac{f(k)}{k} - \delta$$

$$\frac{\dot k}{k} = \frac{sf(k)}{k} - \delta - g - n \qquad \dot k = sf(k) - (\delta + n + g)k$$

$$\frac{\dot k}{k} = 0 \qquad \text{“}k^*\text{”} \qquad sf(k^*) - (\delta + n + g)k^* = 0$$

存在且唯一

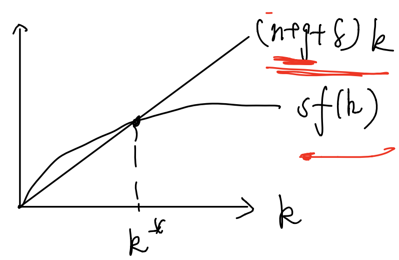
$$\frac{\dot K}{K} = \frac{\dot A}{A} + \frac{\dot L}{L} = n+g$$

$$\frac{\dot Y}{Y} \quad \underline{Y = ALf(k)} \quad \frac{\dot A}{A} + \frac{\dot L}{L} + \frac{\dot k}{k} = n + g + 0 = n+g$$

$$\frac{\dot C}{C} \quad \underline{(1-s)\cdot Y} \quad \frac{\dot{\overline{1-s}}}{1-s} + n + g = n+g$$

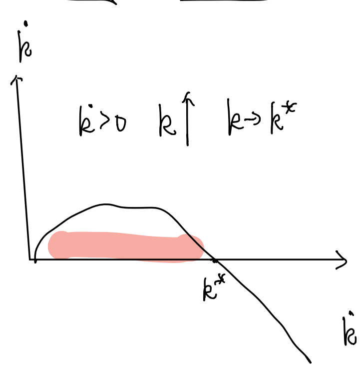
Transitional Dynamics

$$\dot k = \underline{sf(k)} - \underline{(\delta + n + g)k}$$



“持平投资”

实际投资



$\dot k > 0 \quad k\uparrow \quad k \to k^*$

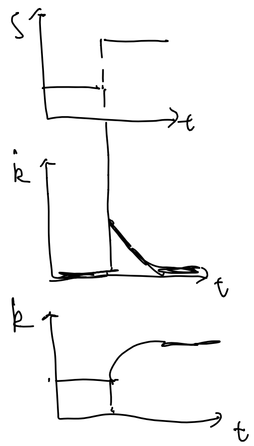
$\dot k < 0 \quad k\downarrow \quad k \to k^*$

k 收敛于 k^*

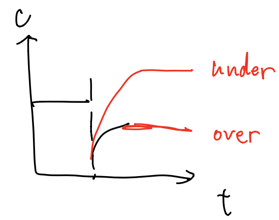
$\Downarrow$

经济收敛于 s.s

"Shock" : $s \uparrow$



$$\boxed{\begin{gathered}\dot{k} = \underline{s} f(k) - (n+g+\delta)k \\ \dot{k}\uparrow \quad \dot{k} \to 0\end{gathered}}$$



$c = (1-s) \cdot Y$

Golden Rule.

max c^* $\longrightarrow k^* \longrightarrow k_{\text{golden Rule}}$

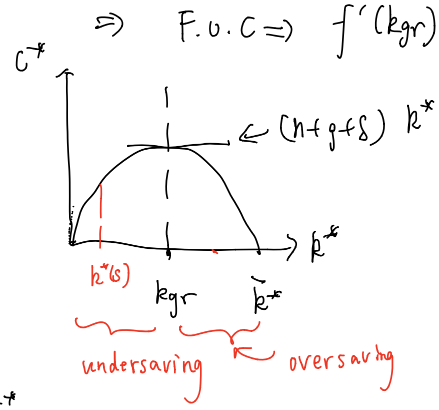
$c = (1-s) f(k) = f(k) - s f(k)$

s.s. $\quad s f(k^*) = (n+g+\delta) k^*$

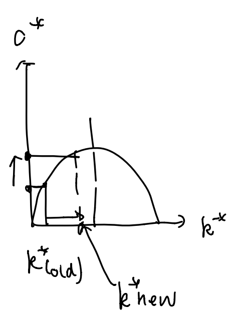
$\Rightarrow c^* = f(k^*) - (n+g+\delta) k^*$

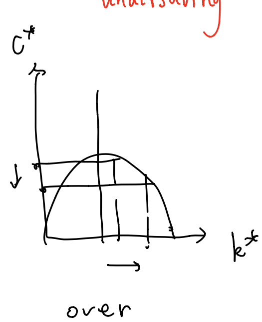
max $f(k^*) - (n+g+\delta) k^*$

$\Rightarrow$ F.O.C $\Rightarrow f'(k_{gr}) = n+g+\delta$



under





$$\boxed{\begin{aligned} A &\to \text{exogenous} \\ s &\to \text{exogenous}\end{aligned}}$$

limitation

高级宏观经济学（二）RCK model

Ramsey – Cass – Koopmans Model

Assumption → Behavior → Equilium Path / S.S → Shock (Dynamics)

A: production function $Y = F(K, AL)$ $\delta = 0$

Endowment $K(0)$ $A(0)$ $L(0)$ → √ $\dot{L}(t) = nL(t)$

Preference
$$U\left\{\{C(t)\}_{t\in[0,\infty)}\right\} = \int_{t=0}^{\infty} e^{-\rho t}\, \boxed{u(C(t))}\, \frac{L(t)}{H}\, dt$$

$L(t)$ → 总人口

ρ 折现率

家庭 (household)

$\frac{L(t)}{H}$: 家庭成员数

$u(C(t))$: 瞬时效用函数

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} \quad \theta > 0$$

CRRA : constant relative risk aversion $\vec{\theta}$

$$u(C(t)) = \frac{C(t)^{1-\theta} - 1}{1-\theta} \xrightarrow{\theta \to 1} \ln C(t)$$

$$u(C(t)) = \begin{cases} \dfrac{C(t)^{1-\theta}}{1-\theta} & 0<\theta<1 \quad 1<\theta<\infty \\ \ln C(t) & \theta \to 1 \end{cases}$$

marginal utility

$$\frac{d\ln(mu(c))}{d\ln c} = \frac{d\ln c^{-\theta}}{d\ln c} = -\theta$$

$c\uparrow$, $mu(c)\downarrow$ θ单位

θ大 → 平滑 θ越小 → 不能避风险.

$r\uparrow$ 替代效应: 当期 cost↑ , $c\downarrow$, $s\uparrow$ θ大, 替小, 风厌.

⇓ 收入效应: 更少储蓄 也 可维持消费 $c\uparrow$, $s\downarrow$

c

market structure : $\begin{cases} \text{Final good} \\ \text{capital} \\ \text{labor} \end{cases}$ 完竞

Behavior Household:

$$(*)\begin{cases}\max\limits_{C(t)\,L(t)} \int_0^\infty e^{-R(t)} u(C(t))\, L(t)\, dt \\ s.t. \int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \le \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} W(t)\, L(t)\, dt \end{cases}$$

消费 ≤ 收入

$\uparrow \frac{L(t)}{H}$

$$\begin{cases} c(t) = \frac{C(t)}{A(t)} & \text{consumption per effective labor} \\ k(t) = \frac{K(t)}{A(t)L(t)} \\ w(t) = \frac{W(t)}{A(t)} & \text{wage per effective labor} \\ A(t) = A(0)e^{gt} \\ L(t) = L(0)e^{nt} \end{cases}$$

$$(*) \Rightarrow \max_{c(t)} B \int_0^\infty e^{-\beta(t)} \frac{c(t)^{1-\theta}}{1-\theta} dt \qquad B = \frac{A(0)^{1-\theta} L(0)}{H}$$

$\Delta\ \beta = \rho - n - (1-\theta)g$

$\beta<0$ 无上界

$\beta>0$ $\rho > n+(1-\theta)g$

$$s.t. \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \le k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$$

$$\mathcal{L} = \underbrace{B \int_{t=0}^{\infty} e^{-\beta(t)} \frac{c(t)^{1-\theta}}{1-\theta} dt}_{} + \lambda\left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt - \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt\right]$$

$\downarrow$ "离散"

$$B \sum_0^\infty e^{-\beta(t)} \frac{c(t)^{1-\theta}}{1-\theta}$$

$t=s$: $$\boxed{\frac{\partial L}{\partial c(s)} = 0 \Rightarrow Be^{-\beta s} c(s)^{-\theta} = \lambda e^{-R(s)} e^{(n+g)s}}$$

$$Be^{-\beta t} c(t)^{-\theta} = \lambda e^{-R(t)} e^{(n+g)t}$$

$\downarrow$ 取对数 $\ln B - \beta t - \theta \ln(c(t)) = \ln\lambda - R(t) + (n+g)t$

$\downarrow$ 对t求导 $$\frac{\dot c(t)}{c(t)} = \frac{r(t) - (n-g-\beta)}{\theta} = \frac{r(t) - \rho - \theta g}{\theta}$$

Firm (max profit)

$$\max_{K(t)\,L(t)} \int_{t=0}^{\infty} e^{-R(t)} [F(K(t), A(t)L(t)] - W(t)L(t) - r(t)K(t)]\, dt$$

$= MP_K = f'(k(t)) = \frac{dF}{dK}$ (rent rate)

$\hookrightarrow MP_L = A(t)[f(k(t)) - k(t)f'(k(t))]$

$$\frac{dR(t)}{dt} = r(t)$$

(Solow)　　　　　　　　储蓄 (Household)

$$\dot{K}(t) = I - \delta K(t) \overset{S=I}{=} I = \underbrace{w(t)A(t)L(t)}_{\text{劳动收入}} + r(t)K(t) - c(t)A(t)L(t)$$

劳动收入 + 资利 − 消 = 储.

$$k(t) = \frac{K(t)}{A(t)L(t)} \qquad \dot{k}(t)/k(t) = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{w(t)}{k(t)} + r(t) - \frac{c(t)}{k(t)} - (n+g)$$

$$\dot{k}(t) = w(t) + r(t)k(t) - c(t) - (n+g)k(t)$$

$$\dot{k}(t) = sf(k(t)) - (n+g)k(t)$$

Firm
$$\begin{cases} r(t) = f'(k(t)) & ① \\ w(t) = f(k(t)) - k(t)f'(k(t)) & ② \end{cases}$$

Household
$$\begin{cases} \dfrac{\dot{c}(t)}{c(t)} = \dfrac{r(t) - \rho - \theta g}{\theta} & ③ \\ \dot{k}(t) = [w(t) + r(t)k(t) - c(t)] - (n+g)k(t) & ④ \end{cases}$$

$\Downarrow$

$$\begin{cases} \dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta} \\ \dot{k}(t) = [f(k(t)) - c(t)] - (n+g)k(t) \end{cases}$$

☆

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(t)}{c(t)} + \frac{\dot{A}(t)}{A(t)} = \frac{r(t) - \rho}{\theta} \quad \uparrow$$

$\theta \uparrow \Rightarrow$ stable

$r(t) \uparrow \Rightarrow$ more cons.

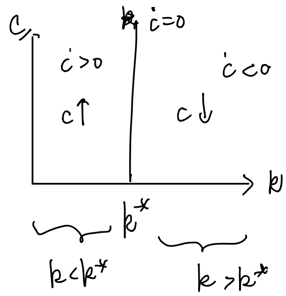
$\rho \uparrow \Rightarrow$ 更耐心 → 平滑 → 储蓄少

Neoclassical Growth

Steady State

(1) $\frac{\dot{c}(t)}{c(t)}=0 \Rightarrow f'(k^*(t)) = \rho+\theta g \qquad f''(k(t)) < 0$

$$\begin{cases} \text{当 } \underline{k>k^*} & f'(k^*) < \rho+\theta g \quad \frac{\dot{c}(t)}{c(t)} < 0 \quad c\downarrow \\ \underline{k<k^*} & f'(k^*) > \rho+\theta g \quad c\uparrow \end{cases}$$

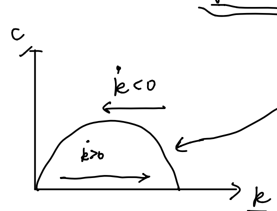


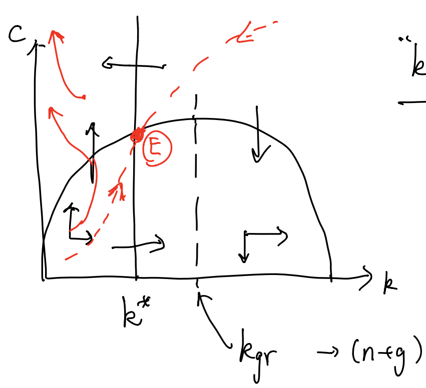
相图

(2) $\dot{k}(t) = [f(k(t)) - c(t)] - (n+g)k(t)$

$\dot{k}(t)=0 \qquad c(t) = f(k^*(t)) - (n+g)k^*(t)$

$$\begin{cases} \dot{k}(t)>0 \quad k\uparrow & \underline{f(k(t)) - (n+g)k(t)} > c(t) \\ \dot{k}(t)<0 \quad k\downarrow & \underline{f(k(t)) - (n+g)k(t)} < c(t) \end{cases}$$





"k_{gr}"

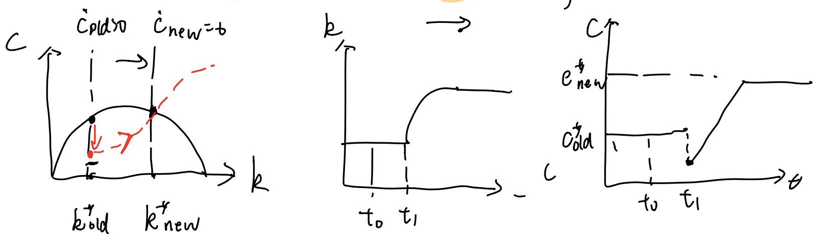
相图.

Shock

(1) unexpected. 永久. $\rho\downarrow$

$$\begin{cases} \dfrac{\dot c(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta} \uparrow \\ \dot k(t) = [f(k(t)) - c(t)] - (n+g)k(t) \end{cases}$$

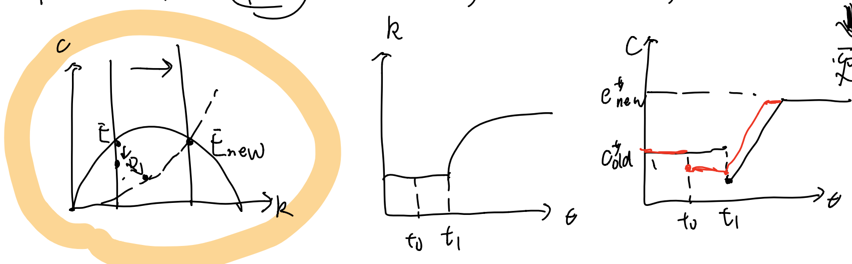
耐心↑ 现消费↓ 将消↑



(2) expected 永久 $\rho\downarrow$ 发生 t_1 ; 预期: t_0 ; 有能力改善行为

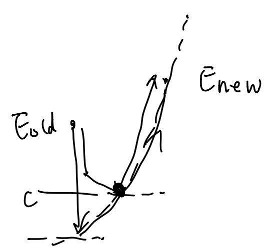
⇓

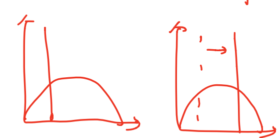
更好.



t_0: $E^*_{old} \rightarrow D$ t_1: F (路径)

$t_0 \rightarrow t_1$ 原先的移动规律





高级宏观经济学(二) OG Model 代际

年轻人 ⟷ 老人

同代 → C_1 C_2

年轻
年老

For 人:

$$\max \frac{(C_{1t})^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{(C_{2t+1})^{1-\theta}}{1-\theta}$$

$$s.t.\quad C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t W_t \quad (*)$$

t时刻机会成本

t时刻打工

{ 年轻打工
只消费

$$L = \frac{(C_{1t})^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{(C_{2t+1})^{1-\theta}}{1-\theta} + \lambda\left[A_t W_t - \left(C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1}\right)\right]$$

$$\frac{\partial L}{\partial C_{1t}} = C_{1t}^{-\theta} - \lambda = 0 \qquad \frac{\partial L}{\partial C_{2t+1}} = \frac{(C_{2t+1})^{-\theta}}{1+\rho} - \frac{\lambda}{1+r_{t+1}} = 0$$

$$\frac{C_{1t}^{-\theta}}{1+r_{t+1}} = \frac{C_{2t+1}^{-\theta}}{1+\rho} \qquad \boxed{C_{2t+1} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} C_{1t}}$$

$$C_{1t} + \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} C_{1t} = A_t W_t$$

$$C_{1t} = A_t W_t \Big/ \left[1 + \frac{(1+\rho)^{-\frac{1}{\theta}}}{(1+r_{t+1})^{-\frac{1}{\theta}}\,(1+r_{t+1})}\right]$$

(r) 替代效应 收入效应

① $\theta=1$ $|sub| = |income|$ $C_{1t} = A_t W_t \cdot (1+\rho)/(2+\rho)$

② $\theta>1$ $|sub| < |income|$ $r_{t+1}\uparrow$ 收大, [减储增消], $C_{1t}\uparrow$

③ $\theta<1$ $|sub| > |income|$ $r_{t+1}\uparrow$ 替大, [增储减消], $C_{1t}\downarrow$

$r\uparrow$ → { sub 第一期消费机会成本大 ⟹ 增储减消
income. 更少储蓄 ⟺ 原先的消费 ⟹ 减储增消

Household.

Saving.

$$S_t^{old} = K_t r_t - K_t(1+r_t) = -K_t < 0.$$

$$S_t^{young} = s(r_{t+1}) A_t W_t L_t$$

$$\dot{K} = \underline{K_{t+1} - K_t} = S_t = S_t^{old} + S_t^{young} = \underline{S_t^{young} - K_t}$$

$$K_{t+1} = S_t^{young}$$

$$k = \frac{K}{AL} \qquad \frac{K_{t+1}}{A_{t+1}L_{t+1}} \quad \frac{A_{t+1} L_{t+1}}{A_t L_t} = k_{t+1}(1+g)(1+n) = \frac{S_t^{young}}{A_t L_t} = s(r_{t+1}) W_t.$$

①

Firm:
$$\begin{cases} r_{t+1} = f'(k_{t+1}) & ② \\ w(t) = f(k(t)) - kf'(k(t)) & ③ \end{cases}$$

$$k_{t+1} = \frac{1}{(1+g)(1+n)} \, sf'(k_{t+1}) \left[f(k(t)) - k_t f'(k(t)) \right].$$

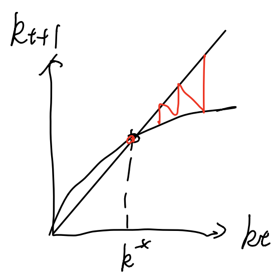
$\theta = 1$

$$f(k) = k^\alpha \qquad f'(k) = \alpha k^{\alpha-1}$$

$$k_{t+1} = \frac{1}{(1+g)(1+n)} \frac{1}{2+\rho} (1-\alpha) k_t^\alpha.$$

s.s. $\quad k_{t+1}^* = k_t^*$

$$k_t^* = \frac{1-\alpha}{(1+g)(1+n)} \frac{1}{2+\rho} \left(\frac{1}{2-\alpha} \right)$$



"k_{gr}"

$$K_{t+1} - K_t = S_t = F[K_t, A_t L_t] - C_t$$

equilium

$\downarrow$ $A_t L_t$

$$\frac{k_{t+1}}{A_t L_t} - k(t) = \frac{S_t}{A_t L_t} = \frac{F}{A_t L_t} - \frac{C_t}{A_t L_t}$$

$$k_{t+1}(1+n)(1+g) - k(t) = \overline{f(k(t))} - c_t$$

$\swarrow$ if α = ... $\qquad \searrow$ if α < 0

$\swarrow$ if $g=0$

$k_{t+1}(1+n) - k(t) = f(k_t) - c_t$

$\downarrow$ s.s.

$$k_{t+1} = k_t = k^*$$

$\downarrow$ $c^* = f(k^*) - nk^*$

$\max_{k^*} c^*$

$\Rightarrow$ $f'(k_{gr}) = n$

$f'(k_{gr}) = n + g + ng$

if $k^* > k_{gr}$ $\Rightarrow$ Oversaving

Shock. unexpected } 一样.
expected

R&D Model

Solow: s (储) 外   RCK OG

A (技) 外

R&D Romer

Assumption

⇒ production function:

R&D sector: $\dot{A}(t) = B[K_R(t)]^{\beta}[L_R(t)]^{\gamma}A(t)^{\theta}$ $\beta \geq 0$ $\gamma \geq 0$ $B > 0$

CRS $\beta + \gamma \neq 1$

$Y = K^{\alpha}L^{\beta}$ $\alpha + \beta = 1$

Product sector: $Y(t) = [K_G(t)]^{\alpha}[A(t)L_G(t)]^{1-\alpha}$ $0 \leq \alpha \leq 1$

⇒ Endowment

$K(0)$ $A(0)$ $L(0)$ ✓

$\dot{L}(t) = nL(t)$

⇒ Preferences

(Solow)

⇒ Market Structure:

α_K:	个人	capital
α_L:	个人	Labor

for free

政府控制 R&D ← 国研究

政府要求征税、 税

Behavior

⇒ individual

$\begin{cases} \text{R\&D} & \text{支付 tax} \\ \boxed{\text{生产} \quad \text{"after-tax" labor} \to \text{labor market} \quad \text{"after-tax" capital} \to \cdots} \end{cases}$

⇒ R&D $\quad K_R(t) = \alpha_K K(t) \quad L_R(t) = \alpha_L L(t)$

$$\dot{A}(t) = B[K_R(t)]^\beta [L_R(t)]^\gamma A(t)^\theta$$

$$\left\{ \begin{array}{l} \dot{A}(t) = B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta \\ \hline B>0 \quad \beta \geq 0 \quad \gamma \geq 0 \end{array} \right.$$

⇒ Goods producing firms

$$\max_{K_G(t)\ L_G(t)} [K_G(t)]^\alpha [A(t) L_G(t)]^{1-\alpha} - W_t L_G(t) - r_t K_G(t)$$

⇒ F.O.Cs: $\quad r(t) = MP_{K_G}$

$\quad W(t) = MP_{L_G}$

Equilium

individuals: $\quad \dot{k} = I - \delta k = s \cdot \text{Total Income} - \delta k \overset{\delta=0}{=\!=} s \cdot \text{Total Income} \quad ①$

Firms : $\quad \text{Total income} = W L_G + r K_G = F(K_G, A L_G) \quad ②$

② → ① $\quad \dot{K} = s F(K_G, AL_G)$

$$= s\,[(1-\alpha_K) K(t)]^\alpha [A(t)(1-\alpha_L) L(t)]^{1-\alpha}$$

Steady State

③ $\quad \dot{k} = s(1-\alpha_K)^\alpha K(t)^\alpha A(t)^{(1-\alpha)} L(t)^{(1-\alpha)} (1-\alpha_L)^{1-\alpha}$

$$= \underbrace{s(1-\alpha_K)^\alpha (1-\alpha_L)^{1-\alpha}}_{C_K} K(t)^\alpha [A(t) L(t)]^{1-\alpha}$$

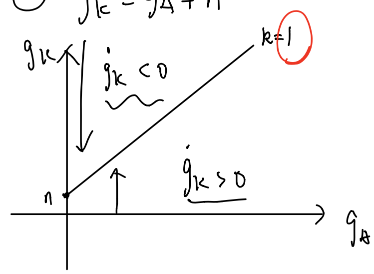
④ $\quad \dot{A}(t) = B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta = \underbrace{B \alpha_K^\beta \alpha_L^\gamma}_{C_A} K(t)^\beta L(t)^\gamma A(t)^\theta$

$$g_K = \frac{\dot{K}}{K} = C_K K(t)^{\alpha-1} [A(t) L(t)]^{1-\alpha}$$

$$\frac{\dot{g}_K}{g_K} = (\alpha-1) g_K + (1-\alpha)(g_A + n)$$

$\Downarrow$ S.S. $\quad \dot{g}_K = 0 \Downarrow \dot{g}_K / g_K = 0$

$\Rightarrow$ ⑤ $g_K = g_A + n$

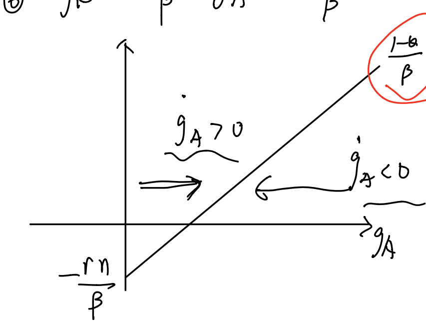


$$g_A \overset{\text{def}}{=} \frac{\dot{A}}{A} = C_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$$

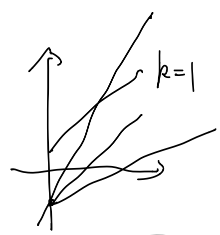
$$\frac{\dot{g}_A}{g_A} = \beta g_K + \gamma n + (\theta-1) g_A$$

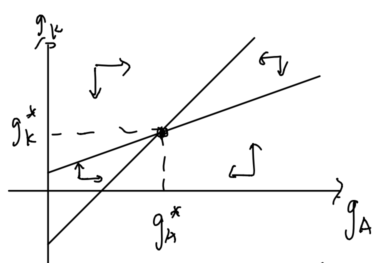
$\Downarrow \quad \dot{g}_A = 0$

$\Rightarrow$ ⑥ $g_K = \frac{1-\theta}{\beta} g_A - \frac{\gamma n}{\beta}$



$\frac{1-\theta}{\beta} > 1 \Rightarrow \beta + \theta < 1$





$$\begin{cases} g_K^* = \frac{1+\gamma-\theta}{1-\theta-\beta} n \\ g_A^* = \frac{\beta+\gamma}{1-\theta-\beta} n \end{cases}$$

high $\underline{K}$

$\beta+\theta < 1$

A $\quad$ K

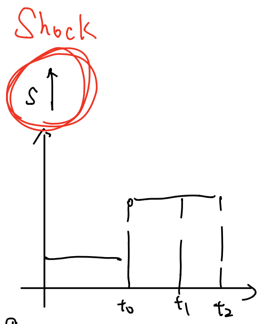
g_K^*

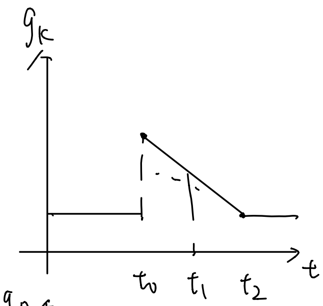
$$Y(t) = [K_G(t)]^{\alpha} [A(t) L_G(t)]^{1-\alpha}$$

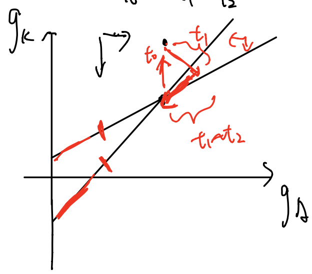
$$g_Y = \frac{\dot{Y}}{Y} = \alpha g_K + (1-\alpha)(g_A + n) \xrightarrow{S.S.} g_Y^* = g_K^*$$

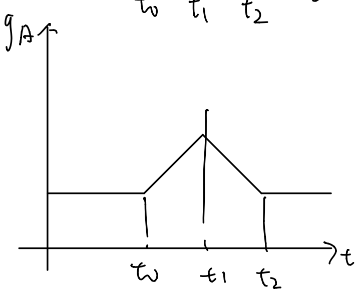
g_K

$\frac{\dot{K}}{K} \quad \frac{\dot{A}}{A} \quad \frac{\dot{L}}{L}$

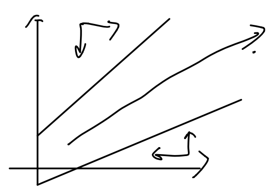








$\beta + \theta > 1$



$\boxed{R\&D}$ $\boxed{\text{intermediate good}}$ Final goods

L_A $\quad L(i,t) \xrightarrow{\Sigma} L_Y(t)$

$$\underbrace{L_A \quad L(i,t)}_{L(t)=L=L_A(t)+L_Y(t)}$$

Final producter: $L(j,t) \to$ min cost.

$$Y(t)=\left[\int_{i=0}^{A(t)} [y(i,t)]^{\phi}\, di\right]^{1/\phi}$$

$$\left[\begin{array}{l} \min\limits_{L(j,t)} \int_{i=1}^{A(t)} p(i,t)\, L(i,t)\, di \\ \text{s.t.} \quad Y(t)=1 \end{array}\right.$$

// $Y(t)=1$

最有效. $y(i,t)=y(j,t)=\dfrac{L_Y(t)}{A(t)}$

$= L(i,t)$

$= L(j,t)$

$$\mathcal{L}=\int_{i=1}^{A(t)} p(i,t)\, L(i,t)\, di + \lambda\left[1-Y(t)\right]$$

$$=\int_{i=1}^{A(t)} p(i,t)\, L(i,t)\, di + \lambda\left\{1-\left[\int_{i=0}^{A(t)} [L(i,t)]^{\phi} di\right]^{1/\phi}\right\}$$

$\lambda \to$ the shadow price of output $\Leftrightarrow P(t)$

$$\frac{\partial \mathcal{L}}{\partial L(j,t)} = p(j,t) - \lambda\, Y(t)^{1-\phi}\, L(j,t)^{\phi-1} = 0$$

$$\overset{Y(t)=1}{=\!=} \quad p(j,t)=\lambda L(j,t)^{\phi-1} \qquad L(j,t)=\left[\frac{\lambda}{p(j,t)}\right]^{\frac{1}{1-\phi}} = \left[\frac{P(t)}{p(j,t)}\right]^{\frac{1}{1-\phi}}$$

$$\overset{Y(t)\neq 1}{=\!=} \quad L(j,t)=\left[\frac{P(t)}{p(j,t)}\right]^{\frac{1}{1-\phi}} \cdot Y(t)$$

转换计价物 $\quad L(j,t)=\left[\dfrac{1}{p(j,t)}\right]^{\frac{1}{1-\phi}} \cdot Y$

$$\frac{d\ln L(j,t)}{d\ln p(j,t)} = \left|-\frac{1}{1-\phi}\right| = \frac{1}{1-\phi}$$

R&D 决定什么?

$P \to y(i,t)$ // $L(i,t)$

$W(t)$

$y=D(p)$

$$\varepsilon_p \overset{\text{def}}{=} \frac{d\ln D}{d\ln p} = \frac{dD/D}{dp/p} = \frac{p}{D}\,\frac{dD}{dp}$$

$$\max \quad p(i,t)\, y(i,t) - W(t)\, y(i,t)$$

$$\Rightarrow \begin{cases} \max \quad py - wy \\ \text{s.t.} \quad y = D(p) \end{cases} \Rightarrow \max \quad pD(p) - wD(p)$$

F.O.C $\quad D(p) + pD'(p) - wD'(p) = 0$

$$\Rightarrow \quad p = w - p \cdot \frac{D(p)}{D'(p) \cdot p} = w - \frac{p}{\varepsilon_p}$$

$$p = \frac{\varepsilon_p}{\varepsilon_p + 1} w \qquad \eta = |\varepsilon_p| = \frac{1}{1-\phi} \qquad p = \frac{\eta}{\eta - 1} w = \frac{1}{\phi} w$$

$$P = \frac{\eta}{\eta - 1} W$$

mark-up

$$p(i,t) = \frac{1}{\phi} W(t)$$

$$\pi(i,t) = p(i,t)\, y(i,t) - W(t)\, y(i,t) = \left(\frac{1}{\phi} - 1\right) W(t)\, y(i,t)$$

$$\dot{A}(t) = B L_A(t) A(t) \quad B > 0$$

$\dot{A}(t) = 1$

$$L_A(t) = \frac{1}{BA(t)}$$

(一次)

研究上的人力投入 $\quad W(t)\, L_A(t) = \dfrac{W(t)}{BA(t)}$

(每期)

$$P(t) = \int_t^\infty \pi(i,\tau)\, e^{-[R(\tau) - R(t)]} d\tau$$

$$R(t) = \int_0^t r(s)\, ds$$

$$\frac{W(t)}{BA(t)} = P(t) = \int_t^\infty \pi(i,\tau)\, e^{R(t) - R(\tau)} d\tau$$

$$\begin{cases} \text{持有一期技术 下一期卖出} & \pi(i,t) + \dot{p}(t) \\ \text{先卖出一期技术，再投资} & r(t)\,p(t) \end{cases} \Rightarrow \underline{r(t) = \frac{\pi(i,t)}{p(t)} + \frac{\dot{p}(t)}{p(t)}}$$

$$X(t) = \frac{p(t)\,A(t)}{L}$$ value of pattern held by <u>each people</u>

For consumer / Labor

$$\begin{cases} \max \int_{t=0}^{\infty} e^{-\rho(t)} \ln C(t)\, dt \\ \text{s.t. } \int_{t=0}^{\infty} e^{-R(t)} c(t)\, dt \le \int_{t=0}^{\infty} e^{-R(t)} W(t)\, dt + X(0) \end{cases}$$

$$L = \int_{t=0}^{\infty} e^{-\rho(t)} \ln C(t)\, dt + \lambda \left[\int_{t=0}^{\infty} e^{-R(t)} W(t)\, dt + X(0) - \int_{t=0}^{\infty} e^{-R(t)} c(t)\, dt \right]$$

$$\frac{\partial L}{\partial C(s)} = e^{-\rho(s)} \frac{1}{C(s)} - \lambda e^{-R(s)} = 0 \qquad -\rho - \frac{\dot{c}}{c} = -r(s)$$

$$-\rho - \frac{\dot{c}(t)}{c(t)} = -r(t)$$

$$\Rightarrow \text{①} \quad \frac{\dot{C}(t)}{C(t)} = r - \rho = \frac{\pi}{p} + \frac{\dot{p}}{p} - \rho$$

$$\Rightarrow \text{②} \quad \dot{X}(t) = \underbrace{W(t) - C(t)}_{\text{Saving}} + r(t)X(t) = B A(t) P(t) - C(t) + r(t) X(t)$$

① $\pi(i,t) = \pi(j,t) = (\frac{1}{\phi} - 1) W(t)\, y(i,t) = (\frac{1}{\phi} - 1) W(t) \frac{L_Y(t)}{A(t)}$

$W(t) = B A(t)\, p(t) \Rightarrow \frac{W(t)}{A(t)} = B p(t)$

$\pi(i,t) = (\frac{1}{\phi} - 1) B p(t) L_Y(t)$ $\Big\}\ \pi$

$\Rightarrow$ ① $\frac{\pi}{p} + \frac{\dot{p}}{p} = (\frac{1}{\phi} - 1) B L_Y(t) + \frac{\dot{p}}{p}$

$$X(t) = \frac{p(t)\,A(t)}{L} \; \text{☆} \quad \Rightarrow \frac{\dot{X}}{X} = \frac{\dot{p}}{p} + \frac{\dot{A}}{A} = \frac{\dot{p}}{p} + B L_A(t)$$

$\frac{\dot{p}}{p} = \frac{\dot{X}}{X} - B L_A(t)$ $\Big\}\ p.$

$$\frac{\dot{C}(t)}{C(t)} = r - \rho = \frac{\pi}{p} + \frac{\dot{p}}{p} - \rho$$

$$\boxed{\frac{\dot{c}(t)}{c(t)} = \frac{1}{\phi} B L_Y(t) - B\bar{L} + \frac{\dot{X}}{X} - \rho}$$

② $\dot{X}(t) = \underbrace{B A(t) P(t)}_{X(t)\bar{L}} - C(t) + r(t) X(t)$

$$= B X(t) \cdot \bar{L} - C(t) + \left[\frac{1}{\phi} B L_Y(t) - B\bar{L} + \frac{\dot{X}}{X}\right] X(t)$$

$$\boxed{\frac{1}{\phi} B L_Y(t) = \frac{C(t)}{X(t)}}$$

$$\begin{cases} \frac{1}{\phi} B L_Y(t) - B\bar{L} + \frac{\dot{X}(t)}{X(t)} - \rho = \frac{\dot{c}(t)}{c(t)} \\ \frac{1}{\phi} B L_Y(t) = \frac{c(t)}{X(t)} \end{cases}$$ Dynamic System

S.S. $\dot{x}(t)$ $\dot{c}(t)$

$$\frac{1}{\phi} B L_Y(t) = B\bar{L} + \rho$$

$$L_Y(t) = \phi\bar{L} + \frac{\phi}{B}\rho$$

$$L_A(t) = \bar{L} - L_Y(t) = (1-\phi)\bar{L} - \frac{\phi}{B}\rho$$

$$\frac{\dot{A}}{A} \overset{\dot{A}(t) = B L_A(t) A(t)}{=\!=\!=} B L_A(t) = B(1-\phi)\bar{L} - \phi\rho$$

$B\uparrow \Rightarrow \frac{\dot{A}}{A}\uparrow$

$\bar{L}\uparrow \Rightarrow \frac{\dot{A}}{A}\uparrow$

$\rho\uparrow \Rightarrow \frac{\dot{A}}{A}\downarrow$

$\phi\uparrow \Rightarrow \frac{\dot{A}}{A}\downarrow$

$\phi\uparrow$ → ① $\pi\downarrow$ → R&D $\Downarrow$

$\phi\uparrow$ → ② $L_A = (1-\phi)\bar{L} - \frac{\phi}{B}\rho$ → R&D $\Downarrow$

$$\frac{\dot{Y}}{Y} \overset{Y(t) = L_Y(t) A(t)^{\frac{1-\phi}{\phi}}}{=\!=\!=} \frac{1-\phi}{\phi}\frac{\dot{A}}{A} = \frac{\dot{c}}{c} = \frac{\dot{X}}{X}$$

$W(t) = B\underline{p(t)}A(t) = B\underline{X(t)}\,\underline{\bar{L}}$ $\qquad \dfrac{\dot{X}}{X} = \dfrac{\dot{w}}{W}$

$\dfrac{\dot{p}}{p} \quad \underline{\underline{p(t) = \dfrac{X(t)}{A(t)}}} \quad \dfrac{\dot{X}(t)}{X(t)} - \dfrac{\dot{A}(t)}{A(t)} = \left(\dfrac{1-\phi}{\phi} - 1\right)\dfrac{\dot{A}}{A} = \dfrac{1-2\phi}{\phi}\dfrac{\dot{A}}{A}$

$\pi = \underline{\left(\frac{1}{\phi} - 1\right)Bp(t)}\ \underline{L_y(t)}$

$\dfrac{\dot{\pi}}{\pi} = \dfrac{\dot{p}}{p} = \dfrac{1-2\phi}{\phi}\dfrac{\dot{A}}{A}$

$p(i,t) = \frac{1}{\phi}W(t)$

$\pi(i,t) = \left(\frac{1}{\phi} - 1\right)W(t)\,y(i,t)$

$$\begin{array}{ccccc} \text{R\&D} & \longrightarrow & \text{intermediate good} & \longrightarrow & \text{final goods} \\ \uparrow L_A & & \uparrow L(i,t) & & Y \\ \text{Labor: } \bar{L} = L_A(t) + L_y(t) & & & & \end{array}$$

/ consumer

$\Rightarrow L(j,t) = \left[\dfrac{1}{p(j,t)}\right]^{\frac{1}{1-\phi}} Y$

$Y(t) = \left[\int_{i=0}^{A(t)} [y(i,t)]^{\phi}\, di\right]^{1/\phi}$

Final producter $\quad \left.\begin{array}{l} \min\limits_{L(j,t)} \int_{i=1}^{A(t)} p(i,t)\,L(i,t)\,di \\ \text{s.t. } Y(t) = 1 \end{array}\right\} \Rightarrow L(j,t) = \left[\dfrac{1}{p(j,t)}\right]^{\frac{1}{1-\phi}} Y$

选择 $L(j,t)$ 最小化成本

R&D sector $\quad \left.\begin{array}{l} \max\limits_{p}\ p(i,t)\,y(i,t) - W(t)\,y(i,t) \\ \text{s.t. } Y = D(p) \end{array}\right\} \Rightarrow \begin{array}{l} p(i,t) = \frac{1}{\phi}W(t) \\ \pi(i,t) = \left(\frac{1}{\phi} - 1\right)W(t)\,y(i,t) \end{array}$

选择 $p(i,t)$ 最大化利润

$\dfrac{W(t)}{B\,A(t)} = \int_t^{\infty} \pi(i,\tau)\, e^{R(t)-R(\tau)}\, d\tau$

$r(t) = \dfrac{\pi(i,t)}{p(t)} + \dfrac{\dot{p}(t)}{p(t)}$

$X(t) = \dfrac{p(t)A(t)}{\bar{L}}$

Labor :

$\begin{cases} \max \int_{t=0}^{\infty} e^{-\rho t} \ln c(t)\,dt \\ \text{s.t. } \int^{\infty} e^{-R(t)} \ldots \int^{\infty} e^{-R(t)} \ldots \end{cases}$

$\int_{t=0} e^{\cdots} c(t)\,dt \le \int_{t=0} e^{-\cdots} W(t)\,dt + W(0)$

$$\frac{\dot{c}}{c} = r - \rho = \frac{\pi}{p} + \frac{\dot{p}}{p} - \rho$$

$$\dot{x} = BA(t)\,p(t) - c(t) + r(t)\,X(t)$$

Dynamic System

$$\begin{cases} \frac{1}{\phi} B L_Y(t) - B\bar{L} + \frac{\dot{X}(t)}{X(t)} - \rho = \frac{\dot{C}(t)}{C(t)} \\ \frac{1}{\phi} B L_Y(t) = \frac{C(t)}{X(t)} \end{cases}$$

S. S

$$\begin{cases} L_Y(t) = \phi \bar{L} + \frac{\phi}{B}\rho \\ L_A(t) = \bar{L} - L_Y(t) = (1-\phi)\bar{L} - \frac{\phi}{B}\rho \end{cases}$$

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\dot{X}}{X} = \frac{\dot{W}}{W} = \frac{1-\phi}{\phi}\frac{\dot{A}}{A}$$

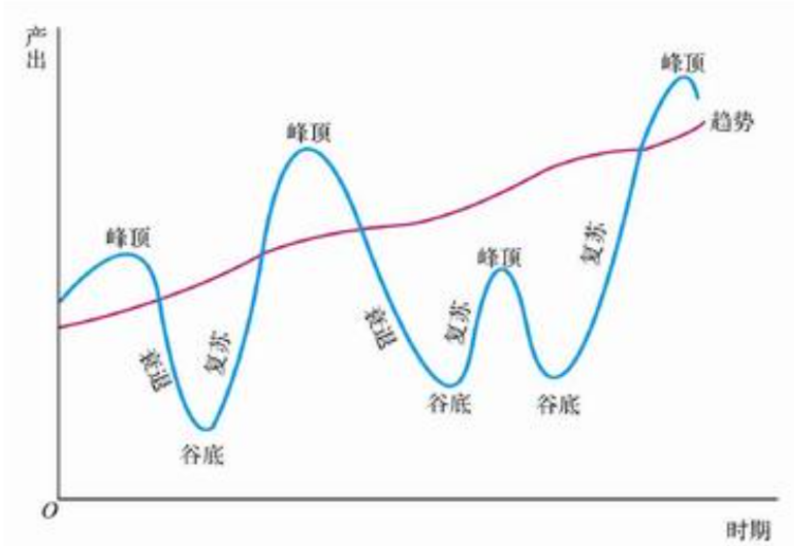
$$\frac{\dot{p}}{p} = \frac{\dot{\pi}}{\pi} = \frac{1-2\phi}{\phi}\frac{\dot{A}}{A}$$

$\frac{1}{1-\phi}$ ϕ ε_p

高级宏观经济学（四）

- Short Fluctuation

Trend



$y_t = \ln(GDP)$

$\bar{y}_t = \ln(\overline{GDP}) = f(k_t^*) \cdot A(t) = f(k_t^*)A(0)e^{gt}$

$\bar{y}_t$表示经济随着时间变动的增长趋势，但是实际经济发展过程y_t中是存在波动的。

用$\tilde{y}_t = y_t - \bar{y}_t$来表示经济波动的大小（实际产出和预期均衡产出之间的差异）

$$\tilde{y}_t = y_t - \bar{y}_t = \ln(GDP) - \ln(\overline{GDP}) = \ln\left(\frac{GDP - \overline{GDP}}{\overline{GDP}} + 1\right) \approx \frac{GDP - \overline{GDP}}{\overline{GDP}}$$

Stylized Facts of Business Cycle

1.Co-movement

1.1 Coefficient

对于任意要素$\widetilde{x}_t$，如果：

- $Cov(\widetilde{y}_t, \widetilde{x}_t) > 0$，称$\widetilde{x}_t$是顺经济周期的（pro-cyclical）
- $Cov(\widetilde{y}_t, \widetilde{x}_t) < 0$，称$\widetilde{x}_t$是逆经济周期的（counter-cyclical）
- $Cov(\widetilde{y}_t, \widetilde{x}_t) = 0$，称$\widetilde{x}_t$是非周期的（a-cyclical）

1.2 Variance

根据波动关系来描述经济要素之间的联动，在实际中，投资，产出和消费波动具有如下经验：

$$Var(I_t) > Var(GDP_t) > Var(C_t)$$

1.3 Timing

- 先于$\widetilde{y}_t$变动的变量，称为leading variable
- 滞后于$\widetilde{y}_t$变动的变量，称为lagging variable
- 与$\widetilde{y}_t$同时发生变动的变量，称为coincident variable

Stylized Facts of Business Cycle

2. Persistence

持续性是指，某个要素的冲击效应可能会持续好几个期间。

在实际中，经济要素的变动一般是以季度为最小期间进行衡量，冲击效应至少持续几个季度以上。
如：在实际中，利率变动导致投资变动， 但是投资者不会马上增加投资，而是逐步增加投资。

经济中的摩擦（friction）导致了这种冲击效应的持续性，进而导致了经济波动的周期性变化，使得冲击效应通常呈现出驼峰状（hump-shaped）。

Stylized Facts of Business Cycle

3. Money

货币在短期不是中性的，会对经济造成影响

Relationship

RCK + shock ⇒ RBC

IS-LM + Micro foundation + shock ⇒ New Keynesian

Note

利率的变动分为内生和外生变动。

内生：生产力的提高本身会带来实际利率的上升
外生：外在力量（如政策）冲击

内生的利率上升是由于增加的产出导致的;
外生的利率上升会减少消费，导致产出下降（在短期内利率变动并不是中性的，利率变动会影响产出）;
因此，如果不能判断利率变动是内生的还是外生的，就不能判断变动对产出的实际影响。
（可以用向量自回归等方法来判断）